Chaotic itinerancy in a 1-d lattice of harmonic potential wells.

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Abstract—
The motion of a damped, driven, particle moving in a periodic harmonic potential having translational invariance is simulated. Traps are observed; the itinerant wandering along the potential well sequence is more complex than a simple Markov process can model.

I. Introduction

Some years ago we reported [1] the investigation of a 1-dimensional two-centre system, consisting of a pair of harmonic potential wells (having linear restoring force/displacement characteristics) with force centres at $\pm X_0$ and a cusp of potential at the origin.

In that system, the motion in each of the potential wells is damped, leading to a contracting mapping, and the motion is driven at a frequency not equal to the natural resonant frequency of each well. That system displays periodic attractors, chaos, and intermittencies or trapping brought on by the presence of grazing [2] bifurcations. That system was studied both by electronic analogue circuit construction and by simulation.

That system is essentially a linear damped second-order harmonic oscillator with a controlled switch which sets the centre of the motion as the distance variable passes the origin. As such, the equations of motion may be solved analytically, and the crossing instants then obtained numerically. The motion reduces to a discrete iteration sequence. It is of importance for the study of chaos in linear systems containing contingent switches, which is an example of what we have called a Variable Structure System [3].

That work is extended here to the case of an unlimited 1-d lattice of identical potential wells. The individual wells are labelled by a "well number" n, and the evolution of n over time is studied by simulation. As yet we have found no easy way of implementing this system by actual analogue circuitry, but the indications from the double-well case are that the simulations give a good guide to the behaviour of the real system.

In this paper we display some preliminary results of running the simulation. The form of $n(t)$ depends on the parameters of the system; namely the amplitude of the drive, the frequency normalised to the natural resonant frequency, the damping factor, and the initial conditions. The time variable t is displayed in terms of the number of cycles of the drive waveform.

II. The system

This system consists of a one-dimensional unending string of harmonic potential wells, in which the local restoring force is linearly related to the displacement from the local centre, and directed towards it. The force constant is shown in figure 1.

In each potential well, the motion is damped and forced. The forcing function is sinusoidal, with a frequency close to, but unrelated to, the natural resonant frequency of an individual well. The differential equation for the system is (assuming unit mass)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + K\frac{\mathrm{d}x}{\mathrm{d}t} + \Omega^2 F(x) = A\sin\omega t \qquad (1)$$

where the periodic restoring force is $-\Omega^2 F(x)$, with $F(x) = x - 2\lfloor(1+x)/2\rfloor$, an odd sawtooth waveform with period equal to two.

The forcing function is arranged to be sufficiently large in amplitude, compared to the damping factor, that the resulting limit cycle lies beyond the bounds of the local potential well. If the system starts at a centre with zero velocity, it spirals (in phase space) out until it falls into one of the two adjacent wells. The velocity changes sign with respect to the direction of the controlling centre, and the forcing function now removes energy from the system instead of adding energy as previously. Consequently, in the adjacent well, the motion spirals in towards the centre before picking up again and the process repeats as before.

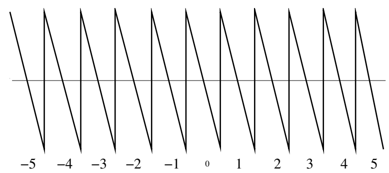

Figure 1: The periodic restoring force against x, well numbers are indicated.

III. Computer experiments

These simulations have been carried out with some care, using the Newton-Raphson method to approximate the crossing points and 64 bit computational precision. The software includes facilities for plotting return maps and Poincaré sections as well as the time development of the well number index $n(t)$.

The results show that there are trapping regions. These may be localised traps, where the motion is periodic, and after a transient (which may be long-lasting) the motion settles down between $n = N$ and $n = N + M$. Alternatively they may be propagating traps, which we call glides, where the trap number n increments uniformly (but not necessarily monotonically) as time increases. In this case the well number increases without limit.

There is also evidence, in the commonly-met chaotic development of $n(t)$, of short term memory about the direction of motion. Thus long runs with a bias towards positive-direction motion may be followed by localisation, and then long runs of negative-direction motion. This behaviour we have attempted to model by a Markov process; in a typical Markov analysis we find that the propensity to keep moving in the same direction may be 0.8, whereas the propensity for reversal may be 0.2. It turns out [5] that a Markov process, while returning the correct statistics for reversals, is far too simple a model for this system.

What is abundantly clear from the observations of the trapping phenomena is that the statistics of the motion of this system are not necessarily stationary. Thus "past performance is no guide to future behaviour" and therefore it is not perhaps surprising that the plots of $n(t)$ are sometimes very reminiscent of stock market and share indices.

A representative selection of the results for the well-number time series $n(t)$ is displayed in figures 2 to 7. Figure 8 shows the Poincaré section (displacement versus velocity at a succession of positive-going zero-crossings of the forcing function) for the time series shown in Figure 2.

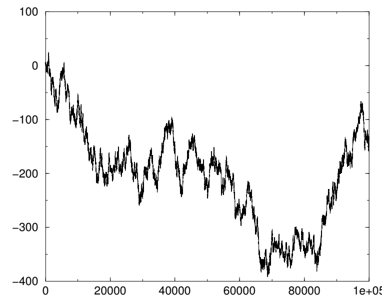

Figure 2: The itinerancy: well number (y-axis) vs drive cycle number

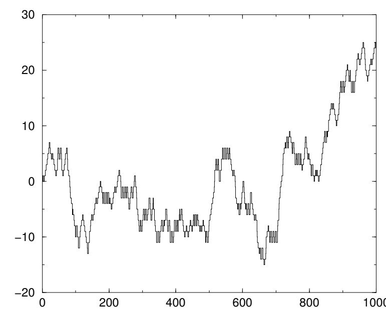

Figure 3: Hesitation.

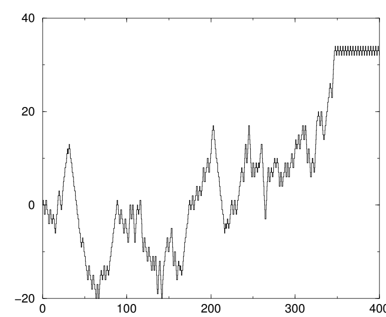

Figure 4: Trapping.

IV. Discussion

One measure of the behaviour of the system is to run repeated simulations to discover the probability of n exceeding some defined distance $\pm M$ within a time

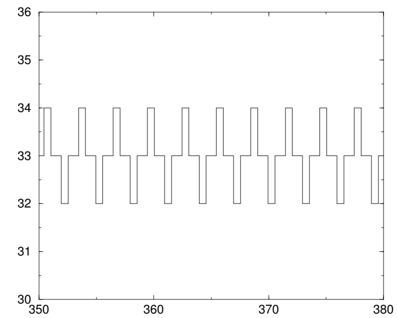

Figure 5: Oscillation in a trap.

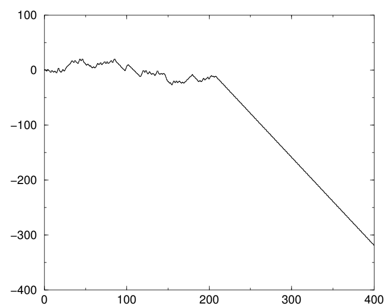

Figure 6: A glide.

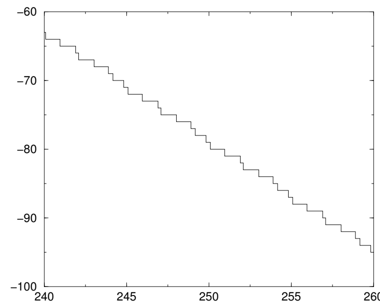

Figure 7: Jagging in the glide.

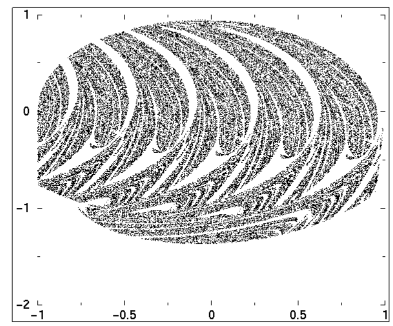

Figure 8: Poincaré section for fig 2.

T. Unfortunately, while this might be done for small M, in general the demands on computing power mean that such a simulation is difficult to perform in such a way as to return adequate statistics. An unanswered question, in the light of the very long chaotic itinerant transients which are sometimes observed, is "Is there a fixed point or trap for this particular combination of parameters?" If one cannot be sure that a trap does not exist, then the statistics of the chaotic transient region may be misleading.

Methods of designing and implementing an analogue circuit version of this system have been considered. Second order systems are commonly implemented by two-integrator-loops; for this system it would be possible to swap the capacitors on the integrators for adjacent wells, at the crossing instance. It is not clear that this could be done with sufficient experimental precision to provide a robust analogue circuit model. Alternatively, a single two-integrator-loop could be used together with a displacement-limit sensor and a switched inverter to transfer the motion leaving one side of the well into motion into the other side. The well numbers could be tracked by a simple counter. Extensions to the 2-d and 3-d case may be considered, although there is a problem with the geometry of the height of the potential cusp at the contact regions in dimensions greater than one. Extensions to the case of cyclic boundary conditions, where M wells are joined in a ring and the value of n modulo M gives a cycle number, may also be made. Suggestions for applications for this case are sought.

Non-uniformity in the potential wells will affect the trapping behaviour. There will also be effects of computational rounding errors and of added noise. Is there a sense in which this system generates its own "noise"? If such is the case, then we might expect the qualitative effects of adding extra noise to be minimal. The rounding error of the computation may be intentionally adjusted, and experiments performed to establish its effect.

The term "chaotic itinerancy" applied to this system seems appropriate, but the itinerancy observed here is somewhat different from the usual use of the term, which seems to imply motion which hops between competing chaotic attractors which may have small regions of overlap. In particular, the propensity to "hesitate" close to a trap before resuming the itinerant progression, and the propensity to fall into long glides before reversing direction, both indicate to these authors that the statistics of finite-span time series samples may not be stationary. In the discussion of the meaning of words in complex systems research, Falconer [4] suggests, in his use of the term "fractal", that it is appropriate to relax the more usual strong mathematical definitions of the technical terms.

Further investigation of this system has revealed that the motion cannot be accurately modelled by a random Markov process, as there are "missing words" in the sequence of left-to-right and right-to-left transitions. This work is reported in the conference [5] Complex Systems 2000, to be held at the University of Otago, New Zealand, in November 2000.

In the course of this preliminary investigation, we examined the form of the well number development $n(t)$ for domains of the variable "number of cycles of the drive" t ranging from 0 to 1,000, to 0 to 1,000,000. It seems a reasonable conjecture that for some specific values of the parameters of drive amplitude, drive frequency, and damping, the motion is self similar in the technical sense. Future work will examine the Hurst parameter for this system.

V. Conclusion

This model is presented as a very appealing example of a simple, basically linear, system, containing contingent switches which demonstrates emergent complexity in its motions. It is particularly easy to simulate, and prompts one to search for possible applications. One of the reasons for bringing this work to NOLTA is to ask for ideas on the applications. This author is reminded of charge-carrier transport properties in certain classes of electronic solids; of the transport properties of communicating processes on a network of computers; and of the migration of sand particles on a corrugated vibrating surface.

Finally, this system may be of interest as a starting point for students who need an interesting but not-too-demanding computational task in the area of complex system simulation.

References

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